The q-deformed mKP hierarchy with self-consistent sources, Wronskian solutions and solitons

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Abstract. Based on the eigenfunction symmetry constraint of the q-deformed modified KP hierarchy, a q-deformed mKP hierarchy with self-consistent sources (q-mKPHSCSs) is constructed. The q-mKPHSCSs contains two types of q-deformed mKP equation with self-consistent sources. By combination of the dressing method and the method of variation of constants, a generalized dressing approach is proposed to solve the q-deformed KP hierarchy with self-consistent sources (q-KPHSCSs). Using the gauge transformation between the q-KPHSCSs and the q-mKPHSCSs, the q-deformed Wronskian solutions for the q-KPHSCSs and the q-mKPHSCSs are obtained. The one-soliton solutions for the q-deformed KP (mKP) equation with a source are given explicitly.

PACS number: 02.30.Ik

1. Introduction

In recent years, the q-deformed integrable systems attracted many interests both in mathematics and in physics (see, e.g., [1-13]). The q-deformation is performed by using the q-derivative ∂_q to take the place of ordinary derivative ∂_x in the classical systems, where q is a parameter, and the q-deformed integrable systems recover the classical ones as $q \to 1$. The q-deformed integrable systems usually inherit some integrable structures from the classical integrable systems. Take the q-deformed KP hierarchy as an example, its τ-function, bi-Hamiltonian structure and additional symmetries have already been reported (see [7, 8, 9, 13] and the references therein).

Multi-component generalization of an integrable model is a very important subject (see, e.g., [14-16]). For example, the multi-component KP (mcKP) hierarchy given in [14] contains many physically relevant nonlinear integrable systems, such as Davey-Stewartson equation, two-dimensional Toda lattice and three-wave resonant interaction ones. Another type of coupled integrable systems is the soliton equation with self-consistent sources, which has many physical applications and can be obtained by coupling some suitable differential equations to the original soliton equation [17-23]. Recently, a systematical procedure is proposed to construct a KP hierarchy with self-consistent sources and its Lax representation [24, 25]. This idea can be used to the q-deformed case, i.e., by introducing a new vector filed ∂_{τ_k} as a linear combination of all vector fields ∂_{t_n} in the q-deformed KP hierarchy, then one can get a new Lax type equation which consists of the τ_k-flow and the evolutions of wave functions. Under the evolutions of wave functions, the commutativity of ∂_{τ_k}-flow and ∂_{t_n}-flows gives rise to a q-KP hierarchy with self-consistent sources (q-KPHSCSs) [26]. The q-KPHSCSs consists of t_n-flow, τ_k-flow, and t_n-evolutions of the eigenfunctions and adjoint eigenfunctions. This q-KPHSCSs contains two types of q-deformed KP equation with self-consistent sources ($1st$-q-KPSCS and $2nd$-q-KPSCS), and the two kinds of reductions of the q-KPHSCSs give the q-deformed Gelfand-Dickey hierarchy with self-consistent sources and the constrained q-deformed KP hierarchy, respectively, which are some (1+1)-dimensional q-deformed soliton equation with self-consistent sources [26].

The dressing method is an important tool for solving the soliton hierarchy [27]. However, this method can not be applied directly to solve the hierarchy with self-consistent sources. In this paper, with the combination of dressing method and the method of variation of constants, a generalized dressing method for solving the q-KPHSCSs is proposed. In this way, we can get some solutions of the q-KPHSCSs in a unified and simple procedure. As a special case, the N-soliton solutions of the two types of q-KPSCS's are obtained simultaneously.

To our knowledge, compared to the study on the q-deformed KP hierarchy, there are less results on the *q-deformed* modified KP hierarchy (q-mKPH). Takasaki studied the q-mKPH and its quasi-classical limit by considering the q-analogue of the tau function of the modified KP hierarhcy [12]. As another part of this paper, we will present the q-mKPH explicitly, and then construct an q-deformed mKP hierarchy with self-consistent sources (q-mKPHSCSs) on the base of eigenfunction symmetry constraint. The q-mKPHSCSs provides a unified way to construct two types of q-deformed modified KP equation with self-consistent sources ($1st$-q-mKPSCS and $2nd$-q-mKPSCS). Then a gauge transformation between the q-KPHSCSs and the q-mKPHSCSs is presented. Since the Wronskian solutions to the q-KPHSCSs have been obtained by a generalized dressing approach in former part of this paper, the gauge transformation enables us to get the explicit formulation of q-deformed Wronskian solutions for the q-mKPHSCSs. It should be noticed that a general setting of "pseudo-differential" operators on regular time scales has been proposed to construct some integrable systems [28, 29], where the q-differential operator is just a particular case.

This paper will be organized as follows. In Sec. 2, we briefly recall how to construct the q-KPHSCSs and its Lax representation, and it is shown that the q-KPHSCSs includes two types of the q-KPSCS. In Sec. 3, a generalized dressing method for the q-KPHSCSs will be proposed. In Sec. 4, a q-deformed mKP hierarchy with self-consistent sources (q-mKPHSCSs) will be constructed, and which includes two types of q-deformed mKP equation with self-consistent sources. In Sec. 5, the gauge transformation between the q-KPHSCSs and the q-mKPHSCSs is established. In Sec. 6, the one-soliton solutions of the q-deformed KP (mKP) equation with a source are obtained. In the last section, some conclusions and remarks will be given.

2. The q-KP hierarchy with self-consistent sources (q-KPHSCSs)

First, we introduce some useful formula for q-KP hierarchy. We denote the q-shift operator and the q-difference operator by θ and ∂_q, respectively, where q is a parameter. These operators act on a function $f(x)$ ($x \in \mathbf{R}$) as

$$\theta(f(x)) = f(qx), \qquad \partial_q(f(x)) = \frac{f(qx) - f(x)}{x(q-1)}.$$

In this paper, we use $P(f)$ to denote an action of a difference operator P on the function f, while $P \circ f = Pf$ means the multiplication of a difference operator P and a zero-th order difference operator f, e.g., $\partial_q f = (\partial_q(f)) + \theta(f)\partial_q$.

Let ∂_q^{-1} denote the formal inverse of ∂_q. In general, the following q-deformed Leibnitz rule holds

$$\partial_q^n f = \sum_{k\geq 0} \begin{pmatrix} n \\ k \end{pmatrix}_q \theta^{n-k}(\partial_q^k f)\partial_q^{n-k}, \qquad n \in \mathbf{Z}, \tag{2.1}$$

where the q-number and the q-binomial are defined by

$$(n)_q = \frac{q^n - 1}{q-1}, \qquad \begin{pmatrix} n \\ k \end{pmatrix}_q = \frac{(n)_q(n-1)_q \cdots (n-k+1)_q}{(1)_q(2)_q \cdots (k)_q}, \qquad \begin{pmatrix} n \\ 0 \end{pmatrix}_q = 1.$$

For a q-pseudo-differential operator (q-PDO) of the form $P = \sum\limits_{i=-\infty}^{n} p_i \partial_q^i$, we decompose P into the differential part $P_+ = \sum\limits_{i=0}^{n} p_i \partial_q^i$ and the integral part $P_- = \sum\limits_{i\leq -1} p_i \partial_q^i$. And the conjugate operation "$*$" for $P = \sum\limits_{i=-\infty}^{n} p_i \partial_q^i$ is defined by $P^* = \sum\limits_{i=-\infty}^{n} (\partial_q^*)^i p_i$ with

$$\partial_q^* = -\partial_q \theta^{-1} = -\frac{1}{q}\partial_{\frac{1}{q}}, \qquad (\partial_q^{-1})^* = (\partial_q^*)^{-1} = -\theta \partial_q^{-1}.$$

The q-exponent $e_q(x)$ is defined as

$$e_q(x) = \exp(\sum_{k=1}^{\infty} \frac{(1-q)^k}{k(1-q^k)} x^k).$$

Then it is easy to prove that $\partial_q^k(e_q(xz)) = z^k e_q(xz)$, $k \in \mathbf{Z}$.

The Lax equation of the q-KP hierarchy (q-KPH) is given by (see, e.g., [7])

$$L_{t_n} = [B_n, L] \equiv B_n L - L B_n, \tag{2.2}$$

where $L = \partial_q + u_0 + u_1 \partial_q^{-1} + u_2 \partial_q^{-2} + \cdots$, $B_n = L_+^n$ stands for the differential part of L^n.

For any fixed $k \in \mathbf{N}$, by introducting a new variable τ_k whose vector field is

$$\partial_{\tau_k} = \partial_{t_k} - \sum_{i=1}^{N} \sum_{s \geq 0} \zeta_i^{-s-1} \partial_{t_s},$$

where ζ_i's are arbitrary distinct non-zero parameters, then a q-deformed KP hierarchy with self-consistent sources (q-KPHSCSs) can be constructed as following [26]

$$L_{\tau_k} = [B_k + \sum_{i=1}^{N} \phi_i \partial_q^{-1} \psi_i, L], \tag{2.3a}$$

$$L_{t_n} = [B_n, L], \qquad \forall n \neq k, \tag{2.3b}$$

$$\phi_{i,t_n} = B_n(\phi_i), \qquad \psi_{i,t_n} = -B_n^*(\psi_i), \qquad i = 1, \cdots, N. \tag{2.3c}$$

The following proposition is proved in [26].

Proposition 1 *The commutativity of (2.3a) and (2.3b) under (2.3c) gives rise to the following zero-curvature representation for q-KPHSCSs (2.3)*

$$B_{n,\tau_k} - (B_k + \sum_{i=1}^{N} \phi_i \partial_q^{-1} \psi_i)_{t_n} + [B_n, B_k + \sum_{i=1}^{N} \phi_i \partial_q^{-1} \psi_i] = 0, \tag{2.4a}$$

$$\phi_{i,t_n} = B_n(\phi_i), \qquad \psi_{i,t_n} = -B_n^*(\psi_i), \qquad i = 1, 2, \cdots, N, \tag{2.4b}$$

with the Lax representation given by

$$\Psi_{t_n} = B_n(\Psi), \quad \Psi_{\tau_k} = (B_k + \sum_{i=1}^{N} \phi_i \partial_q^{-1} \psi_i)(\Psi). \tag{2.5}$$

Two kinds of reductions of the q-KPHSCSs (2.3) are also studied in [26], these reductions give many (1+1)-dimensional q-deformed soliton equation with self-consistent sources, e.g., the q-deformed KdV equation with sources, the q-deformed Boussinesq equation with sources.

For convenience,we write out some operators here

$$B_1 = \partial_q + u_0, \qquad B_2 = \partial_q^2 + v_1 \partial_q + v_0, \qquad B_3 = \partial_q^3 + s_2 \partial_q^2 + s_1 \partial_q + s_0,$$

$$\phi_i\partial_q^{-1}\psi_i = r_{i1}\partial_q^{-1} + r_{i2}\partial_q^{-2} + r_{i3}\partial_q^{-3} + \ldots, \qquad i = 1, \ldots, N,$$

where

$$v_1 = \theta(u_0) + u_0, \qquad v_0 = (\partial_q u_0) + \theta(u_1) + u_0^2 + u_1,$$
$$v_{-1} = (\partial_q u_1) + \theta(u_2) + u_0u_1 + u_1\theta^{-1}(u_0) + u_2,$$

$$s_2 = \theta(v_1) + u_0, \qquad s_1 = (\partial_q v_1) + \theta(v_0) + u_0v_1 + u_1,$$
$$s_0 = (\partial_q v_0) + \theta(v_{-1}) + u_0v_0 + u_1\theta^{-1}(v_1) + u_2.$$

$$r_{i1} = \phi_i\theta^{-1}(\psi_i), \qquad r_{i2} = -\frac{1}{q}\phi_i\theta^{-2}(\partial_q\psi_i), \qquad r_{i3} = \frac{1}{q^3}\phi_i\theta^{-3}(\partial_q^2\psi_i).$$

and v_{-1} comes from $L^2 = B_2 + v_{-1}\partial_q^{-1} + v_{-2}\partial_q^{-2} + \cdots$.

Then, one can compute the following commutators

$$[B_2, B_3] = f_2\partial_q^2 + f_1\partial_q + f_0, \qquad [B_2, \phi_i\partial_q^{-1}\psi_i] = g_{i1}\partial_q + g_{i0} + \ldots,$$
$$[B_3, \phi_i\partial_q^{-1}\psi_i] = h_{i2}\partial_q^2 + h_{i1}\partial_q + h_{i0} + \ldots, \qquad i = 1, \ldots, N,$$

where

$$f_2 = \partial_q^2 s_2 + (q+1)\theta(\partial_q s_1) + \theta^2(s_0) + v_1\partial_q s_2 + v_1\theta(s_1) + v_0s_2 - (q^2+q+1)\theta(\partial_q^2 v_1)$$
$$-(q^2+q+1)\theta^2(\partial_q v_0) - (q+1)s_2\theta(\partial_q v_1) - s_2\theta^2(v_0) - s_1\theta(v_1) - s_0,$$

$$f_1 = \partial_q^2 s_1 + (q+1)\theta(\partial_q s_0) + v_1\partial_q s_1 + v_1\theta(s_0) + v_0s_1 - \partial_q^3 v_1 - (q^2+q+1)\theta(\partial_q^2 v_0)$$
$$-s_2\partial_q^2 v_1 - (q+1)s_2\theta(\partial_q v_0) - s_1\partial_q v_1 - s_1\theta(v_0) - s_0v_1,$$

$$f_0 = \partial_q^2 s_0 + v_1\partial_q s_0 - \partial_q^3 v_0 - s_2\partial_q^2 v_0 - s_1\partial_q v_0,$$

$$g_{i1} = \theta^2(r_{i1}) - r_{i1}, \qquad g_{i0} = (q+1)\theta(\partial_q r_{i1}) + \theta^2(r_{i2}) + v_1\theta(r_{i1}) - r_{i1}\theta^{-1}(v_1) - r_{i2},$$

$$h_{i2} = \theta^3(r_{i1}) - r_{i1}, \qquad h_{i1} = (q^2+q+1)\theta^2(\partial_q r_{i1}) + \theta^3(r_{i2}) + s_2\theta^2(r_{i1}) - r_{i1}\theta^{-1}(s_2).$$

$$h_{i0} = (q^2+q+1)\theta(\partial_q^2 r_{i1}) + (q^2+q+1)\theta^2(\partial_q r_{i2}) + \theta^3(r_{i3}) + (q+1)s_2\theta(\partial_q r_{i1})$$
$$+s_2\theta^2(r_{i2}) + s_1\theta(r_{i1}) - r_{i1}\theta^{-1}(s_1) + \frac{1}{q}r_{i1}\theta^{-2}(\partial_q s_2) - r_{i2}\theta^{-2}(s_2) - r_{i3}.$$

Now, we list some examples in the q-KPHSCSs (2.4) [26].

Example 1 *The first type of q-deformed KP equation with self-consistent sources (1st-q-KPSCS) is given by (2.4) with $n = 2$ and $k = 3$*

$$-\frac{\partial s_2}{\partial t_2} + f_2 = 0, \tag{2.6a}$$

$$\frac{\partial v_1}{\partial \tau_3} - \frac{\partial s_1}{\partial t_2} + f_1 + \sum_{i=1}^{N} g_{i1} = 0, \tag{2.6b}$$

$$\frac{\partial v_0}{\partial \tau_3} - \frac{\partial s_0}{\partial t_2} + f_0 + \sum_{i=1}^{N} g_{i0} = 0, \tag{2.6c}$$

$$\phi_{i,t_2} = B_2(\phi_i), \quad \psi_{i,t_2} = -B_2^*(\psi_i), \quad i = 1, 2, \cdots, N. \tag{2.6d}$$

Let $q \to 1$ and $u_0 \equiv 0$, then the 1st-q-KPSCS reduces to the first type of KP equation with self-consistent sources [17, 18, 26].

Example 2 *The second type of q-deformed KP equation with self-consistent sources (2nd-q-KPSCS) is given by (2.4) with $n=3$ and $k=2$*

$$\frac{\partial s_2}{\partial \tau_2} - f_2 + \sum_{i=1}^{N} h_{i2} = 0, \tag{2.7a}$$

$$\frac{\partial s_1}{\partial \tau_2} - \frac{\partial v_1}{\partial t_3} - f_1 + \sum_{i=1}^{N} h_{i1} = 0, \tag{2.7b}$$

$$\frac{\partial s_0}{\partial \tau_2} - \frac{\partial v_0}{\partial t_3} - f_0 + \sum_{i=1}^{N} h_{i0} = 0, \tag{2.7c}$$

$$\phi_{i,t_3} = B_3(\phi_i), \quad \psi_{i,t_3} = -B_3^*(\psi_i), \quad i = 1, 2, \cdots, N. \tag{2.7d}$$

Let $q \to 1$ and $u_0 \equiv 0$, then the 2nd-q-KPSCS reduces to the second type of KP equation with self-consistent sources [17, 26].

3. Generalized dressing approach for the q-KPHSCSs

We will first give the dressing approach for the q-KPH (2.2). Assume that the operator L of q-KP hierarchy (2.2) can be written as a dressing form

$$L = S\partial_q S^{-1}, \tag{3.1}$$

with $S = \partial_q^N + w_1\partial_q^{N-1} + w_2\partial_q^{N-2} + \cdots + w_N$.

It is easy to verify that if S satisfies the Sato equation

$$S_{t_n} = -L_-^n S, \tag{3.2}$$

then L defined by (3.1) satisfies the q-KP hierarchy (2.2).

If there are N linearly independent functions $h_1, \ldots, h_N$ solving $S(h_i) = 0$, then $w_1, \ldots, w_N$ are completely determined by solving the linear equations

$$\begin{pmatrix} h_1 & \partial_q(h_1) & \cdots & \partial_q^{N-1}(h_1) \\ h_2 & \partial_q(h_2) & \cdots & \partial_q^{N-1}(h_2) \\ \vdots & \vdots & \vdots & \vdots \\ h_N & \partial_q(h_N) & \cdots & \partial_q^{N-1}(h_N) \end{pmatrix} \begin{pmatrix} w_N \\ w_{N-1} \\ \vdots \\ w_1 \end{pmatrix} = - \begin{pmatrix} \partial_q^N(h_1) \\ \partial_q^N(h_2) \\ \vdots \\ \partial_q^N(h_N) \end{pmatrix}.$$

Then the operator S can be written as

$$S = \frac{1}{\mathrm{Wrd}(h_1, \cdots, h_N)} \begin{vmatrix} h_1 & h_2 & \cdots & h_N & 1 \\ \partial_q(h_1) & \partial_q(h_2) & \cdots & \partial_q(h_N) & \partial_q \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \partial_q^N(h_1) & \partial_q^N(h_2) & \cdots & \partial_q^N(h_N) & \partial_q^N \end{vmatrix}, \tag{3.3}$$

where

$$\mathrm{Wrd}(h_1, \cdots, h_N) = \begin{vmatrix} h_1 & h_2 & \cdots & h_N \\ \partial_q(h_1) & \partial_q(h_2) & \cdots & \partial_q(h_N) \\ \vdots & \vdots & \vdots & \vdots \\ \partial_q^{N-1}(h_1) & \partial_q^{N-1}(h_2) & \cdots & \partial_q^{N-1}(h_N) \end{vmatrix}.$$

Remark 1 *The denominator of S (3.3) is actually a q-deformed Wronskian determinant, so we may denote it as* $\mathrm{Wrd}(h_1, \cdots, h_N)$. *The numerator of S (3.3) is a formal determinant, which is denoted by* $\mathrm{Wrd}(h_1, \cdots, h_N, \partial_q)$. *It is understood as an expansion with respect to its last column, in which all sub-determinants are collected on the left of the difference operator* ∂_q^j.

Then, we have the dressing approach for the q-KP hierarchy (2.2) as the following.

Proposition 2 *Assume that h_i satisfies*

$$h_{i,t_n} = \partial_q^n(h_i), \qquad i = 1, \cdots, N, \tag{3.4}$$

and S and L are constructed as (3.3) and (3.1) respectively, then S and L satisfy the Sato equation (3.2) and the q-KP hierarchy (2.2).

Proof Apply partial derivative ∂_{t_n} to the equation $S(h_i) = 0$, and notice that h_i's are linearly independent, then we have $S_{t_n} + L_-^n S = 0$. This completes the proof.

Unfortunately, the dressing approach given above can not provide the evolution with respect to the new variable τ_k. Now we will generalize the dressing approach to solve the q-KPHSCSs (2.3) and give exact formulas for ϕ_i and ψ_i. First, we have the following lemma.

Lemma 1 *For any q-pseudo-operator S, if S satisfies*

$$S_{t_n} = -L_-^n S \tag{3.5a}$$

$$S_{\tau_k} = -L_-^k S + \sum_{i=1}^{N} \phi_i \partial_q^{-1} \psi_i S. \tag{3.5b}$$

then L defined by (3.1) satisfies (2.3a) and (2.3b).

Proof We only write out the proof for (2.3a).

$$L_{\tau_k} = S_{\tau_k}\partial_q S^{-1} - S\partial_q S^{-1} S_{\tau_k} S^{-1} = (-L_-^k + \sum_i \phi_i \partial_q^{-1}\psi_i)L + L(L_-^k - \sum_i \phi_i \partial_q^{-1}\psi_i)$$
$$= [-L_-^k + \sum_i \phi_i \partial_q^{-1}\psi_i, L] = [B_k + \sum_{i=1}^{N} \phi_i \partial_q^{-1}\psi_i, L].$$

This dressing operator S can be constructed as following. Let f_i and g_i satisfy

$$f_{i,t_n} = \partial_q^n(f_i), \quad f_{i,\tau_k} = \partial_q^k(f_i) \tag{3.6a}$$

$$g_{i,t_n} = \partial_q^n(g_i), \quad g_{i,\tau_k} = \partial_q^k(g_i), \quad i = 1, \ldots, N. \tag{3.6b}$$

And let h_i be the linear combination of f_i and g_i as

$$h_i = f_i + \alpha_i(\tau_k) g_i, \qquad i = 1, \ldots, N, \tag{3.7}$$

with the coefficient α_i being a differentiable function of τ_k. Suppose $h_1, \ldots, h_N$ are linearly independent. Then clearly S defined by (3.3) and (3.7) still satisfy (3.2) according to Proposition 2. To claim that S satisfy (3.5b), we present

$$\phi_i = -\dot{\alpha}_i S(g_i), \qquad \psi_i = (-1)^{N-i}\theta\left(\frac{\mathrm{Wrd}(h_1, \cdots, \hat{h}_i, \cdots, h_N)}{\mathrm{Wrd}(h_1, \cdots, h_N)}\right), \tag{3.8}$$

where the hat $\hat{}$ means to rule out this term from the q-deformed Wronskian determinant, and $\dot{\alpha}_i = \frac{d\alpha_i}{d\tau_k}$. Then we have the generalized dressing approach for the q-KPHSCSs (2.3) as the following proposition.

Proposition 3 *Let S be defined by (3.3) and (3.7), $L = S\partial_q S^{-1}$, ϕ_i and ψ_i be given by (3.8), then S satisfies (3.5) and L, ϕ_i, ψ_i satisfy the q-KPHSCSs (2.3).*

To prove Proposition 3, we need several lemmas. The first one is the q-deformed version of Oevel and Strampp's lemma [30].

Lemma 2 *(Oevel and Strampp [30]) Under the condition of Proposition 3, we have*

$$S^{-1} = \sum_{i=1}^{N} h_i \partial_q^{-1} \psi_i.$$

Lemma 3 *Under the condition of Proposition 3, we have $\partial_q^{-1} S^*(\psi_i) = 0$, for $i = 1, \ldots, N$.*

Proof It can be proved that

$$(\partial_q^{-1}\psi_i S)_- = \partial_q^{-1} S^*(\psi_i). \tag{3.9}$$

Using Lemma 2, we have

$$\begin{aligned} 0 = (\partial_q^j S^{-1} \circ S)_- &= (\partial_q^j \sum_{i=1}^{N} h_i \partial_q^{-1}\psi_i S)_- = (\sum_{i=1}^{N} \partial_q^j(h_i)\partial_q^{-1}\psi_i S)_- \\ &= \sum_{i=1}^{N} \partial_q^j(h_i)\partial_q^{-1} S^*(\psi_i), \quad j = 0, 1, 2, \ldots. \end{aligned}$$

Solving the equations with respect to $\partial_q^{-1} S^*(\psi_i)$, we get Lemma 3.

Lemma 4 *Under the condition of Proposition 3, the operator $\partial_q^{-1}\psi_i S$ is a non-negative difference operator and*

$$(\partial_q^{-1}\psi_i S)(h_j) = \delta_{ij}, \quad 1 \le i, j \le N. \tag{3.10}$$

Proof Lemma 3 and (3.9) implies that $\partial_q^{-1}\phi_i S$ is a non-negative difference operator. We define functions $c_{ij} = (\partial_q^{-1}\psi_i S)(h_j)$, then $\partial_q(c_{ij}) = \psi_i S(h_j) = 0$, which means c_{ij} does not depend on x in the sense of q-deformed version. From Lemma 2, we find that

$$\begin{aligned} \sum_{i=1}^{N} \partial_q^k(h_i)c_{ij} = \sum_{i=1}^{N} \partial_q^k(h_i c_{ij}) = \partial_q^k(\sum_{i=1}^{N} h_i c_{ij}) &= \partial_q^k(\sum_{i} h_i \partial_q^{-1}\psi_i S(h_j)) \\ &= \partial_q^k(S^{-1}S)(h_j) = \partial_q^k(h_j), \end{aligned}$$

since the functions $h_1, h_2, \cdots, h_N$ are linearly independent, we can easily conclude that $c_{ij} = \delta_{ij}$.

Proof of Proposition 3.

Analogous to the proof of Proposition 2, we can prove (3.5a). For (3.5b), taking ∂_{τ_k} to the identity $S(h_i)=0$,we find

$$0=(S_{\tau_k})(h_i)+(S\partial_q^k)(h_i)+\dot{\alpha}_i S(g_i)=(S_{\tau_k})(h_i)+(L^k S)(h_i)-\sum_{j=1}^N \phi_j\delta_{ji}$$
$$=(S_{\tau_k}+L_-^k S-\sum_{j=1}^N \phi_j\partial_q^{-1}\psi_j S)(h_i).$$

Obviously, $S_{\tau_k}+L_-^k S$ is a pure difference operator of degree $<N$. and moreover using Lemma 4, $\sum_{j=1}^N \phi_j\partial_q^{-1}\psi_j S$ is also a pure difference operator of degree $<N$. so $S_{\tau_k}+L_-^k S-\sum_{j=1}^N \phi_j\partial_q^{-1}\psi_j S$ is a pure difference operator of degree $<N$. Since the non-negative difference operator acting on h_i in the last expression has degree $<N$, it can not annihilate N independent functions unless the operator itself vanishes.

Hence (3.5) is proved. Then Lemma 1 leads to (2.3a) and (2.3b).

The proof of the first equation in (2.3c) is the following.

$$\phi_{i,t_n}=-\dot{\alpha}_i(S(g_i))_{t_n}=-\dot{\alpha}(S_{t_n}+S\partial_q^n)(g_i)$$
$$=-\dot{\alpha}_i(-L_-^n S+L^n S)(g_i)=-\dot{\alpha}B_n S(g_i)=B_n(\phi_i).$$

And it remains to prove the second equation in (2.3c). Firstly, we see that

$$(S^{-1})_{t_n}=((S^{-1})_{t_n})_-=(-S^{-1}S_{t_n}S^{-1})_-=(S^{-1}(L^n-B_n))_-$$
$$=(\partial_q^n S^{-1})_- -(S^{-1}B_n)_-=(\partial_q^n\sum_{i=1}^N h_i\partial_q^{-1}\psi_i)_- -(\sum_{i=1}^N h_i\partial_q^{-1}\psi_i B_n)_-$$
$$=\sum_{i=1}^N \partial_q^n(h_i)\partial_q^{-1}\psi_i-\sum_{i=1}^N h_i\partial_q^{-1}B_n^*(\psi_i).$$

On the other hand, $(S^{-1})_{t_n}=(\sum_{i=1}^N h_i\partial_q^{-1}\psi_i)_{t_n}=\sum_{i=1}^N \partial_q^n(h_i)\partial_q^{-1}\psi_i-\sum_{i=1}^N h_i\partial_q^{-1}\psi_{i,t_n}$, so we have $\sum_{i=1}^N h_i\partial_q^{-1}(B_n^*(\psi_i)+\psi_{i,t_n})=0$. Since h_i, $i=1,...,N$ are linearly independent, it is easy to get $\psi_{i,t_n}=-B_n^*(\psi_i)$.

Thus, we proved Proposition 3 (the generalized dressing approach for the q-KPHSCSs (2.3)).

4. The q-mKP hierarchy with self-consistent sources (q-mKPHSCSs)

In this section, we will construct the q-deformed mKP hierarchy with self-consistent sources (q-mKPHSCSs). The Lax operator $\widetilde{L}$ of q-mKP hierarchy is defined by

$$\widetilde{L}=\widetilde{u}\partial_q+\widetilde{u}_0+\widetilde{u}_1\partial_q^{-1}+\widetilde{u}_2\partial_q^{-2}+\cdots.$$

And the Lax equation of q-mKP hierarchy is given by

$$\widetilde{L}_{t_n}=[\widetilde{B}_n,\widetilde{L}],\qquad \widetilde{B}_n=(\widetilde{L}^n)_{\geq 1}. \tag{4.1}$$

The ∂_{t_n}-flows are commutative with each other, and we can easily deduce the zero-curvature equation

$$\widetilde{B}_{n,t_m} - \widetilde{B}_{m,t_n} + [\widetilde{B}_n, \widetilde{B}_m] = 0. \tag{4.2}$$

When $n = 2$ and $m = 3$, we get the q-mKP equation. If we take $q \to 1$ and $\widetilde{u} \equiv 1$, then the q-mKP equation will reduce to the mKP equation

$$4v_t - v_{xxx} + 6v^2 v_x - 3(D^{-1}v_{yy}) - 6v_x(D^{-1}v_y) = 0,$$

where $t := t_3$, $y := t_2$, $v := \widetilde{u}_0$.

According to the squared eigenfunction symmetry (see [31, 32] and the references therein), we can construct a q-mKP hierarchy with self-consistent sources (q-mKPHSCSs) as

$$\begin{aligned}
\widetilde{L}_{\tau_k} &= [\widetilde{B}_k + \sum_{i=1}^{N} \widetilde{\phi}_i \partial_q^{-1} \widetilde{\psi}_i \partial_q, \widetilde{L}], && (4.3a)\\
\widetilde{L}_{t_n} &= [\widetilde{B}_n, \widetilde{L}], \quad \forall n \neq k, && (4.3b)\\
\widetilde{\phi}_{i,t_n} &= \widetilde{B}_n(\widetilde{\phi}_i), && (4.3c)\\
\widetilde{\psi}_{i,t_n} &= -(\partial_q \widetilde{B}_n \partial_q^{-1})^*(\widetilde{\psi}_i),\ i = 1, \cdots, N. && (4.3d)
\end{aligned}$$

Then it is easy to get the zero curvature equation for the q-mKPHSCSs (4.3)

$$\widetilde{B}_{n,\tau_k} - (\widetilde{B}_k + \sum_{i=1}^{N} \widetilde{\phi}_i \partial_q^{-1} \widetilde{\psi}_i \partial_q)_{t_n} + [\widetilde{B}_n, \widetilde{B}_k + \sum_{i=1}^{N} \widetilde{\phi}_i \partial_q^{-1} \widetilde{\psi}_i \partial_q] = 0. \tag{4.4}$$

Under the condition (4.3c) and (4.3d), the Lax pair for the q-mKPHSCSs (4.3) is given by

$$\Psi_{t_n} = \widetilde{B}_n(\Psi), \qquad \Psi_{\tau_k} = (\widetilde{B}_k + \sum_{i=1}^{N} \widetilde{\phi}_i \partial_q^{-1} \widetilde{\psi}_i \partial_q)(\Psi).$$

First, for convenience, we write out some operators here

$$\begin{aligned}
&\widetilde{B}_1 = \widetilde{u}\partial_q, \qquad \widetilde{B}_2 = \widetilde{v}_2 \partial_q^2 + \widetilde{v}_1 \partial_q, \qquad \widetilde{B}_3 = \widetilde{s}_3 \partial_q^3 + \widetilde{s}_2 \partial_q^2 + \widetilde{s}_1 \partial_q,\\
&\widetilde{\phi}_i \partial_q^{-1} \widetilde{\psi}_i \partial_q = \widetilde{r}_{i0} + \widetilde{r}_{i1} \partial_q^{-1} + \widetilde{r}_{i2} \partial_q^{-2} + \cdots, \qquad i = 1, \ldots, N,
\end{aligned}$$

where

$$\begin{aligned}
&\widetilde{v}_2 = \widetilde{u}\theta(\widetilde{u}), \qquad \widetilde{v}_1 = \widetilde{u}(\theta(\widetilde{u}_0) + \widetilde{u}_0 + \partial_q(\widetilde{u})),\\
&\widetilde{v}_0 = \widetilde{u}_1 \theta^{-1}(\widetilde{u}) + \widetilde{u}_0^2 + \widetilde{u}\theta(\widetilde{u}_1) + \widetilde{u}\partial_q(\widetilde{u}_0),\\
&\widetilde{s}_3 = \widetilde{u}\theta(\widetilde{v}_2), \qquad \widetilde{s}_2 = \widetilde{u}\partial_q(\widetilde{v}_2) + \widetilde{u}\theta(\widetilde{v}_1) + \widetilde{u}_0 \widetilde{v}_2,\\
&\widetilde{s}_1 = \widetilde{u}\partial_q(\widetilde{v}_1) + \widetilde{u}\theta(\widetilde{v}_0) + \widetilde{u}_0 \widetilde{v}_1 + \widetilde{u}_1 \theta^{-1}(\widetilde{v}_2),\\
&\widetilde{r}_{i0} = \widetilde{\phi}_i \theta^{-1}(\widetilde{\psi}_i), \qquad \widetilde{r}_{i1} = -\frac{1}{q}\widetilde{\phi}_i \theta^{-2}(\partial_q \widetilde{\psi}_i), \qquad \widetilde{r}_{i2} = \frac{1}{q^3}\widetilde{\phi}_i \theta^{-3}(\partial_q^2 \widetilde{\psi}_i),
\end{aligned}$$

and $\widetilde{v}_0$ comes from $\widetilde{L}^2 = \widetilde{B}_2 + \widetilde{v}_0 + \widetilde{v}_{-1}\partial_q^{-1} + \cdots$.

Then, one can compute the following commutators

$$\begin{aligned}
&[\widetilde{B}_2, \widetilde{B}_3] = \widetilde{f}_3 \partial_q^3 + \widetilde{f}_2 \partial_q^2 + \widetilde{f}_1 \partial_q, \qquad [\widetilde{B}_2, \widetilde{\phi}_i \partial_q^{-1} \widetilde{\psi}_i \partial_q] = \widetilde{g}_{i2} \partial_q^2 + \widetilde{g}_{i1} \partial_q + \cdots,\\
&[\widetilde{B}_3, \widetilde{\phi}_i \partial_q^{-1} \widetilde{\psi}_i \partial_q] = \widetilde{h}_{i3} \partial_q^3 + \widetilde{h}_{i2} \partial_q^2 + \widetilde{h}_{i1} \partial_q + \cdots, \qquad i = 1, \ldots, N.
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{f}_3 &= \widetilde{v}_2\partial_q^2(\widetilde{s}_3) + (q+1)\widetilde{v}_2\theta(\partial_q(\widetilde{s}_2)) + \widetilde{v}_2\theta^2(\widetilde{s}_1) + \widetilde{v}_1\partial_q(\widetilde{s}_3) + \widetilde{v}_1\theta(\widetilde{s}_2) - (q^2+q+1)\widetilde{s}_3\theta(\partial_q^2(\widetilde{v}_2)) \\
&\quad -(q^2+q+1)\widetilde{s}_3\theta^2(\partial_q(\widetilde{v}_1)) - (q+1)\widetilde{s}_2\theta(\partial_q(\widetilde{v}_2)) - \widetilde{s}_2\theta^2(\widetilde{v}_1) - \widetilde{s}_1\theta(\widetilde{v}_2), \\
\widetilde{f}_2 &= \widetilde{v}_2\partial_q^2(\widetilde{s}_2) + (q+1)\widetilde{v}_2\theta(\partial_q(\widetilde{s}_1)) + \widetilde{v}_1\partial_q(\widetilde{s}_2) + \widetilde{v}_1\theta(\widetilde{s}_1) - \widetilde{s}_3\partial_q^3(\widetilde{v}_2) \\
&\quad -(q^2+q+1)\widetilde{s}_3\theta(\partial_q^2(\widetilde{v}_1)) - \widetilde{s}_2\partial_q^2(\widetilde{v}_2) - (q+1)\widetilde{s}_2\theta(\partial_q(\widetilde{v}_1)) - \widetilde{s}_1\partial_q(\widetilde{v}_2) - \widetilde{s}_1\theta(\widetilde{v}_1), \\
\widetilde{f}_1 &= \widetilde{v}_2\partial_q^2(\widetilde{s}_1) + \widetilde{v}_1\partial_q(\widetilde{s}_1) - \widetilde{s}_3\partial_q^3(\widetilde{v}_1) - \widetilde{s}_2\partial_q^2(\widetilde{v}_1) - \widetilde{s}_1\partial_q(\widetilde{v}_1), \\
\widetilde{g}_{i2} &= \widetilde{v}_2[\theta^2(\widetilde{r}_{i0}) - \widetilde{r}_{i0}], \\
\widetilde{g}_{i1} &= (q+1)\widetilde{v}_2\theta(\partial_q(\widetilde{r}_{i0})) + \widetilde{v}_2\theta^2(\widetilde{r}_{i1}) + \widetilde{v}_1\theta(\widetilde{r}_{i0}) - \widetilde{r}_{i0}\widetilde{v}_1 - \widetilde{r}_{i1}\theta^{-1}(\widetilde{v}_2), \\
\widetilde{h}_{i3} &= \widetilde{s}_3[\theta^3(\widetilde{r}_{i0}) - \widetilde{r}_{i0}], \\
\widetilde{h}_{i2} &= (q^2+q+1)\widetilde{s}_3\theta^2(\partial_q(\widetilde{r}_{i0})) + \widetilde{s}_3\theta^3(\widetilde{r}_{i1}) + \widetilde{s}_2\theta^2(\widetilde{r}_{i0}) - \widetilde{r}_{i0}\widetilde{s}_2 - \widetilde{r}_{i1}\theta^{-1}(\widetilde{s}_3), \\
\widetilde{h}_{i1} &= (q^2+q+1)\widetilde{s}_3\theta(\partial_q^2(\widetilde{r}_{i0})) + (q^2+q+1)\widetilde{s}_3\theta^2(\partial_q(\widetilde{r}_{i1})) + \widetilde{s}_3\theta^3(\widetilde{r}_{i2}) + (q+1)\widetilde{s}_2\theta(\partial_q(\widetilde{r}_{i0})) \\
&\quad +\widetilde{s}_2\theta^2(\widetilde{r}_{i1}) + \widetilde{s}_1\theta(\widetilde{r}_{i0}) - \widetilde{r}_{i0}\widetilde{s}_1 + \frac{1}{q}\widetilde{r}_{i1}\theta^{-2}(\partial_q(\widetilde{s}_3)) - \widetilde{r}_{i1}\theta^{-1}(\widetilde{s}_2) - \widetilde{r}_{i2}\theta^{-2}(\widetilde{s}_3).
\end{aligned}$$

Now, we can list the two types of q-mKP equations with self-consistent source.

Example 3 *When $n=2$ and $k=3$, the q-mKPHSCSs (4.3) gives the first type of q-mKP equation with self-consistent sources (1st-q-mKPSCS)*

$$-\frac{\partial \widetilde{s}_3}{\partial t_2} + \widetilde{f}_3 = 0, \tag{4.5a}$$

$$\frac{\partial \widetilde{v}_2}{\partial \tau_3} - \frac{\partial \widetilde{s}_2}{\partial t_2} + \widetilde{f}_2 + \sum_{i=1}^{N}\widetilde{g}_{i2} = 0, \tag{4.5b}$$

$$\frac{\partial \widetilde{v}_1}{\partial \tau_3} - \frac{\partial \widetilde{s}_1}{\partial t_2} + \widetilde{f}_1 + \sum_{i=1}^{N}\widetilde{g}_{i1} = 0, \tag{4.5c}$$

$$\widetilde{\phi}_{i,t_2} = \widetilde{B}_2(\widetilde{\phi}_i), \quad \widetilde{\psi}_{i,t_2} = -(\partial_q\widetilde{B}_2\partial_q^{-1})^*(\widetilde{\psi}_i), \quad i=1,2,\ldots,N. \tag{4.5d}$$

Let $q\to 1$ and $u\equiv 1$, then the first type of q-mKP equation with self-consistent source (4.5) reduces to the first type of mKP equation with self-consistent sources which reads as

$$\begin{aligned}
&4\widetilde{u}_{0,t} - \widetilde{u}_{0,xxx} + 6\widetilde{u}_0^2\widetilde{u}_{0,x} - 3D^{-1}\widetilde{u}_{0,yy} - 6\widetilde{u}_{0,x}D^{-1}\widetilde{u}_{0,y} + 4\sum_{i=1}^{N}(\widetilde{\phi}_i\widetilde{\psi}_i)_x = 0, \\
&\widetilde{\phi}_{i,y} = \widetilde{\phi}_{i,xx} + 2\widetilde{u}_0\widetilde{\phi}_{i,x}, \\
&\widetilde{\psi}_{i,y} = -\widetilde{\psi}_{i,xx} + 2\widetilde{u}_0\widetilde{\psi}_{i,x}, \qquad i=1,\ldots,N,
\end{aligned}$$

where $t:=\tau_3, y:=t_2$.

Example 4 *When $n=3$ and $k=2$, the q-mKPHSCSs (4.3) gives the second type of q-mKP equation with self-consistent source (2nd-q-mKPSCS)*

$$\frac{\partial \widetilde{s}_3}{\partial \tau_2} - \widetilde{f}_3 + \sum_{i=1}^{N}\widetilde{h}_{i3} = 0, \tag{4.7a}$$

$$\frac{\partial \widetilde{s}_2}{\partial \tau_2} - \frac{\partial \widetilde{v}_2}{\partial t_3} - \widetilde{f}_2 + \sum_{i=1}^{N}\widetilde{h}_{i2} = 0, \tag{4.7b}$$

$$\frac{\partial \widetilde{s}_1}{\partial \tau_2}-\frac{\partial \widetilde{v}_1}{\partial t_3}-\widetilde{f}_1+\sum_{i=1}^{N}\widetilde{h}_{i1}=0, \tag{4.7c}$$

$$\widetilde{\phi}_{i,t_3}=\widetilde{B}_3(\widetilde{\phi}_i),\quad \widetilde{\psi}_{i,t_3}=-(\partial_q\widetilde{B}_3\partial_q^{-1})^*(\widetilde{\psi}_i),\qquad i=1,\cdots,N. \tag{4.7d}$$

Let $q\to 1$ and $u\equiv 1$, then the second type of q-mKP equation with self-consistent source (4.7) reduces to the second type of mKP equation with self-consistent sources which reads as

$$\begin{aligned}&4\widetilde{u}_{0,t}-\widetilde{u}_{0,xxx}+6\widetilde{u}_0^2\widetilde{u}_{0,x}-3D^{-1}\widetilde{u}_{0,yy}-6\widetilde{u}_{0,x}D^{-1}\widetilde{u}_{0,y}\\&\qquad+\sum_{i=1}^{N}[3(\widetilde{\phi}_i\widetilde{\psi}_{i,xx}-\widetilde{\phi}_{i,xx}\widetilde{\psi}_i)-3(\widetilde{\phi}_i\widetilde{\psi}_i)_y-6(\widetilde{u}_0\widetilde{\phi}_i\widetilde{\psi}_i)_x]=0,\end{aligned}$$

$$\widetilde{\phi}_{i,t}=\widetilde{\phi}_{i,xxx}+3\widetilde{u}_0\widetilde{\phi}_{i,xx}+\frac{3}{2}(D^{-1}\widetilde{u}_{0,y})\widetilde{\phi}_{i,x}+\frac{3}{2}\widetilde{u}_{0,x}\widetilde{\phi}_{i,x}+\frac{3}{2}\widetilde{u}_0^2\widetilde{\phi}_{i,x}+\frac{3}{2}\sum_{j=1}^{N}(\widetilde{\phi}_j\widetilde{\psi}_j)\widetilde{\phi}_{i,x},$$

$$\widetilde{\psi}_{i,t}=\widetilde{\psi}_{i,xxx}-3\widetilde{u}_0\widetilde{\psi}_{i,xx}+\frac{3}{2}(D^{-1}\widetilde{u}_{0,y})\widetilde{\psi}_{i,x}-\frac{3}{2}\widetilde{u}_{0,x}\widetilde{\psi}_{i,x}+\frac{3}{2}\widetilde{u}_0^2\widetilde{\psi}_{i,x}+\frac{3}{2}\sum_{j=1}^{N}(\widetilde{\phi}_j\widetilde{\psi}_j)\widetilde{\psi}_{i,x},$$

where $y:=\tau_2, t:=t_3$.

5. The gauge transformation between the q-KPHSCSs and the q-mKPHSCSs

In this section, we will give a gauge transformation between the q-KPHSCSs and the q-mKPHSCSs.

Proposition 4 *Suppose L, ϕ_i's, and ψ_i's satisfy the q-KPHSCSs (2.3), and f is a particular eigenfunction for the Lax pair (2.5) of the q-KPHSCSs, i.e.,*

$$f_{t_n}=B_n(f),\qquad f_{\tau_k}=(B_k+\sum_{i=1}^{N}\phi_i\partial_q^{-1}\psi_i)(f),$$

then

$$\widetilde{L}:=f^{-1}Lf,\quad \widetilde{\phi_i}:=f^{-1}\phi_i,\quad \widetilde{\psi_i}:=-\theta\partial_q^{-1}(f\psi_i)=(\partial_q^{-1})^*(f\psi_i), \tag{5.1}$$

satisfy the q-mKPHSCSs (4.3).

Proof Since f is the eigenfunction of the Lax pair (2.5) for the q-KPHSCSs, then

$$\begin{aligned}\widetilde{L}_{t_n}&=(f^{-1}Lf)_{t_n}=-f^{-1}B_n(f)f^{-1}Lf+f^{-1}[B_n,L]f+f^{-1}LB_n(f)\\&=-f^{-1}B_n(f)\widetilde{L}+[f^{-1}B_nf,\widetilde{L}]+\widetilde{L}f^{-1}B_n(f)=[f^{-1}B_nf-f^{-1}B_n(f),\widetilde{L}]=[\widetilde{B}_n,\widetilde{L}],\end{aligned}$$

here it is used that $\Delta:=f^{-1}B_nf-f^{-1}B_n(f)=f^{-1}[(L^nf)_{\geq 0}-(L^n)_{\geq 0}(f)]=f^{-1}((L^nf)_{\geq 1})=(f^{-1}L^nf)_{\geq 1}=\widetilde{L}^n_{\geq 1}$, and we denote $\widetilde{L}^n_{\geq 1}$ by $\widetilde{B}_n$. Moreover, we have

$$\begin{aligned}\widetilde{L}_{\tau_k}=(f^{-1}Lf)_{\tau_k}&=-f^{-1}(B_k+\sum_{i=1}^{N}\phi_i\partial_q^{-1}\psi_i)(f)f^{-1}Lf+f^{-1}[B_k+\sum_{i=1}^{N}\phi_i\partial_q^{-1}\psi_i,L]f\\&\quad+f^{-1}L(B_k+\sum_{i=1}^{N}\phi_i\partial_q^{-1}\psi_i)(f)\end{aligned}$$

$$
\begin{aligned}
&= [f^{-1}(B_k + \sum_{i=1}^{N} \phi_i \partial_q^{-1} \psi_i) f - f^{-1}(B_k + \sum_{i=1}^{N} \phi_i \partial_q^{-1} \psi_i)(f), \widetilde{L}] \\
&= [\widetilde{B}_k, \widetilde{L}] + \sum_{i=1}^{N} [f^{-1} \phi_i \partial_q^{-1} \psi_i f - f^{-1} \phi_i \partial_q^{-1} \psi_i(f), \widetilde{L}] \\
&= [\widetilde{B}_k, \widetilde{L}] + \sum_{i=1}^{N} [\widetilde{\phi}_i \partial_q^{-1} \circ \partial_q^*(\widetilde{\psi}_i) + \widetilde{\phi}_i \partial_q^{-1} \circ \partial_q \circ \theta^{-1}(\widetilde{\psi}_i), \widetilde{L}] \\
&= [\widetilde{B}_k, \widetilde{L}] + \sum_{i=1}^{N} [\widetilde{\phi}_i \partial_q^{-1} \circ \widetilde{\psi}_i \partial_q, \widetilde{L}] = [\widetilde{B}_k + \sum_{i=1}^{N} \widetilde{\phi}_i \partial_q^{-1} \widetilde{\psi}_i \partial_q, \widetilde{L}],
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\phi}_{i,t_n} &= -f^{-1} B_n(f) f^{-1} \phi_i + f^{-1} B_n(\phi_i) = -f^{-1} B_n(f) \widetilde{\phi}_i + f^{-1} B_n(f \widetilde{\phi}_i) \\
&= f^{-1} L^n_{\geq 0} f(\widetilde{\phi}_i) - f^{-1} L^n_{\geq 0}(f) \widetilde{\phi}_i = (f^{-1} L^n f)_{\geq 0}(\widetilde{\phi}_i) - f^{-1} L^n_{\geq 0}(f) \widetilde{\phi}_i \\
&= (f^{-1} L^n f)_{\geq 1}(\widetilde{\phi}_i) + (f^{-1} L^n_{\geq 0}(f))(\widetilde{\phi}_i) - f^{-1} L^n_{\geq 0}(f) \widetilde{\phi}_i = \widetilde{B}_n(\widetilde{\phi}_i),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\psi}_{i,t_n} &= (\partial_q^{-1})^* [B_n(f) \psi_i - f B_n^*(\psi_i)] = (\partial_q^{-1})^* [B_n(f) f^{-1} \partial_q^*(\widetilde{\psi}_i) - f B_n^* f^{-1} \partial_q^*(\widetilde{\psi}_i)] \\
&= -(\partial_q^{-1})^* [((f (L^n)^* f^{-1})_{\geq 0} \partial_q^*(\widetilde{\psi}_i) - (L^n)_{\geq 0}(f) f^{-1} \partial_q^*(\widetilde{\psi}_i)] \\
&= -(\partial_q^{-1})^* [((f^{-1} L^n f)_{\geq 0})^* \partial_q^*(\widetilde{\psi}_i) - (L^n)_{\geq 0}(f) f^{-1} \partial_q^*(\widetilde{\psi}_i)] \\
&= -(\partial_q^{-1})^* ((f^{-1} L^n f)_{\geq 1})^* \partial_q^*(\widetilde{\psi}_i) = -(\partial_q^{-1})^* \widetilde{B}_n^* \partial_q^*(\widetilde{\psi}_i) = -(\partial_q \widetilde{B}_n \partial_q^{-1})^*(\widetilde{\psi}_i).
\end{aligned}
$$

This completes the proof.

Therefore, if a special eigenfunction f for the Lax pair (2.5) of the q-KPHSCSs is given, then we can get a solution of the q-mKPHSCSs by the gauge transformation (5.1). Here we choose

$$
f = S(1) = (-1)^N \frac{\mathrm{Wrd}(\partial_q(h_1), \partial_q(h_2), \cdots, \partial_q(h_N))}{\mathrm{Wrd}(h_1, h_2, \cdots, h_N)} \tag{5.2}
$$

as the particular eigenfunction for the Lax pair (2.5) of the q-KPHSCSs, where S is the dressing operator defined by (3.3) and (3.7). Then the Wronskian solution for the q-mKPHSCSs is

$$
\widetilde{L} = f^{-1} L f = \frac{\mathrm{Wrd}(h_1, \cdots, h_N, \partial_q)}{\mathrm{Wrd}(\partial_q(h_1), \cdots, \partial_q(h_N))} \partial_q \left[\frac{\mathrm{Wrd}(h_1, \cdots, h_N, \partial_q)}{\mathrm{Wrd}(\partial_q(h_1), \cdots, \partial_q(h_N))} \right]^{-1}, \tag{5.3a}
$$

$$
\widetilde{\phi}_i = f^{-1} \phi_i = -\dot{\alpha}_i \frac{\mathrm{Wrd}(h_1, h_2, \cdots, h_N, g_i)}{\mathrm{Wrd}(\partial_q(h_1), \partial_q(h_2), \cdots, \partial_q(h_N))}, \tag{5.3b}
$$

$$
\widetilde{\psi}_i = -\theta \partial_q^{-1}(f \psi_i) = \theta \left(\frac{\mathrm{Wrd}(\partial_q(h_1), \cdots, \partial_q(\hat{h}_i), \cdots, \partial_q(h_N))}{\mathrm{Wrd}(h_1, h_2, \cdots, h_N)} \right), \qquad i = 1, \ldots, N. \tag{5.3c}
$$

The above expressions for $\widetilde{L}$ and $\widetilde{\phi}_i$'s can be easily known by straightforward calculation, and the above expressions for $\widetilde{\psi}_i$'s can be derived as follows. First, we see

$$
\sum_{i=1}^{N} \theta(h_i) \widetilde{\psi}_i = \sum_{i=1}^{N} \theta(-h_i \partial_q^{-1}(\psi_i f)) = \theta((\sum_{i=1}^{N} -h_i \partial_q^{-1} \psi_i)(f)) = \theta(S^{-1} S(1)) = 1.
$$

And moreover we have the following relation (for $k \geq 1$),

$$
\begin{aligned}
&\sum_{i=1}^{N} \theta(\partial_q^k(h_i))\widetilde{\psi}_i = \sum_{i=1}^{N} \theta[-\partial_q^k(h_i)\cdot \partial_q^{-1}(\psi_i f)] \\
&= \sum_{i=1}^{N} \theta[-\partial_q(\partial_q^{k-1}(h_i)\cdot\partial_q^{-1}(\psi_i f)) + \theta(\partial_q^{k-1}(h_i)\cdot\psi_i f)] \\
&= \cdots = \sum_{i=1}^{N} \theta[-\partial_q^k(h_i\partial_q^{-1}\psi_i(f)) + \sum_{j=0}^{k-1} \partial_q^{k-j-1}(\theta(\partial_q^j(h_i))\psi_i f)] \\
&= \theta[\sum_{j=0}^{k-1} \partial_q^{k-j-1}(\sum_{i=0}^{N} \theta(\partial_q^j(h_i))\psi_i f)].
\end{aligned}
$$

Notice the definition of ψ_i's (3.8) and $\sum_{i=0}^{N} \theta(\partial_q^j(h_i))\psi_i = \delta_{j,N-1}$, $j = 0, 1, \ldots, N-1$, then we have $\sum_{i=1}^{N} \theta(\partial_q^k(h_i))\widetilde{\psi}_i = 0$, for $k = 1, \ldots, N-1$. Then using the Cramer principle, we can get the exact form of $\widetilde{\psi}_i$'s (5.3).

6. Solutions of the q-KPHSCSs and the q-mKPHSCSs

The generalized dressing approach (Proposition 3) and the gauge transformation (5.1) give us a simple way to construct explicit solutions of the q-KPHSCSs and the q-mKPHSCSs. Here we use the first type of q-KP equation with self-consistent sources (2.6) and the first type of q-mKP equation with self-consistent sources (4.5) as the examples.

If we choose

$$
\begin{aligned}
f_i &:= e_q(\lambda_i x)\exp(\lambda_i^2 t_2 + \lambda_i^3 \tau_3) \equiv e_q(\lambda_i x)e^{\xi_i}, \\
g_i &:= e_q(\mu_i x)\exp(\mu_i^2 t_2 + \mu_i^3 \tau_3) \equiv e_q(\mu_i x)e^{\eta_i}, \\
h_i &:= f_i + \alpha_i(\tau_3)g_i = e_q(\lambda_i x)e^{\xi_i} + \alpha_i(\tau_3)e_q(\mu_i x)e^{\eta_i}, \qquad i = 1, \ldots, N,
\end{aligned}
$$

then the generalized dressing approach (Proposition 3) enables us to get the soliton solutions to the first type of q-KP equation with sources (2.6).

Example 5 *(One-soliton solution to the 1st-q-KPSCS (2.6)) Let $N = 1$, then*

$$S = \partial_q + w_0, \quad w_0 = -\frac{\partial_q(h_1)}{h_1}.$$

Notice that $LS = S\partial_q$, i.e., $(\partial_q + u_0 + u_1\partial_q^{-1} + \cdots)(\partial_q + w_0) = (\partial_q + w_0)\partial_q$, then the generalized dressing approach (Proposition 3) gives the one-soliton solution to the first type of q-KP equation with one source ((2.6) with $N = 1$)

$$
\begin{aligned}
u_0 &= (1-\theta)(w_0) = (\theta - 1)(\frac{\partial_q(h_1)}{h_1}) \\
&= \frac{\lambda_1 e_q(\lambda_1 q x)e^{\xi_1} + \alpha_1(\tau_3)\mu_1 e_q(\mu_1 q x)e^{\eta_1}}{e_q(\lambda_1 q x)e^{\xi_1} + \alpha_1(\tau_3)e_q(\mu_1 q x)e^{\eta_1}} - \frac{\lambda_1 e_q(\lambda_1 x)e^{\xi_1} + \alpha_1(\tau_3)\mu_1 e_q(\mu_1 x)e^{\eta_1}}{e_q(\lambda_1 x)e^{\xi_1} + \alpha_1(\tau_3)e_q(\mu_1 x)e^{\eta_1}},
\end{aligned}
$$

$$
\begin{aligned}
u_1 &= -[\partial_q(w_0)+(1-\theta)(w_0)w_0] = \frac{\partial_q^2 h_1}{h_1} - \left(\frac{\partial_q h_1}{h_1}\right)^2 \\
&= \frac{\lambda_1^2 e_q(\lambda_1 x)e^{\xi_1}+\alpha_1\mu_1^2 e_q(\mu_1 x)e^{\eta_1}}{e_q(\lambda_1 x)e^{\xi_1}+\alpha_1 e_q(\mu_1 x)e^{\eta_1}} - \left(\frac{\lambda_1 e_q(\lambda_1 x)e^{\xi_1}+\alpha_1\mu_1 e_q(\mu_1 x)e^{\eta_1}}{e_q(\lambda_1 x)e^{\xi_1}+\alpha_1 e_q(\mu_1 x)e^{\eta_1}}\right)^2,
\end{aligned}
$$

$$
u_2 = -u_1\theta^{-1}(w_0) = u_1\theta^{-1}(\frac{\partial_q(h_1)}{h_1}) = u_1\frac{\lambda_1 e_q(\lambda_1 q^{-1}x)e^{\xi_1}+\alpha_1\mu_1 e_q(\mu_1 q^{-1}x)e^{\eta_1}}{e_q(\lambda_1 q^{-1}x)e^{\xi_1}+\alpha_1 e_q(\mu_1 q^{-1}x)e^{\eta_1}},
$$

$$
\phi_1 = -\dot{\alpha}_1\frac{h_1\partial_q(g_1)-\partial_q(h_1)g_1}{h_1} = -\frac{d\alpha_1}{d\tau_3}e_q(\mu_1 x)e^{\eta_1}[\mu_1 - \frac{\lambda_1 e_q(\lambda_1 x)e^{\xi_1}+\alpha_1(\tau_3)\mu_1 e_q(\mu_1 x)e^{\eta_1}}{e_q(\lambda_1 x)e^{\xi_1}+\alpha_1(\tau_3)e_q(\mu_1 x)e^{\eta_1}}],
$$

$$
\psi_1 = \theta(\frac{1}{h_1}) = \frac{1}{e_q(\lambda_1 qx)e^{\xi_1}+\alpha_1(\tau_3)e_q(\mu_1 qx)e^{\eta_1}}.
$$

Example 6 *(One-soliton solution to the 1st-q-mKPSCS (4.5)) Let $N = 1$, then by gauge transformation, the formulae (5.3) gives*

$$
\widetilde{L} = \tilde{u}\partial_q + \tilde{u}_0 + \tilde{u}_1\partial_q^{-1} + \cdots = (w_1\partial_q - 1)\partial_q(w_1\partial_q - 1)^{-1}, \qquad w_1 = \frac{h_1}{\partial_q(h_1)},
$$

this enables us to get the one-soliton solution to the first type of q-mKP equation with a source ((4.5) with $N = 1$)

$$
\begin{aligned}
\tilde{u} &= \frac{w_1}{\theta(w_1)} = \frac{h_1\theta(\partial_q h_1)}{\partial_q(h_1)\theta(h_1)} \\
&= \frac{(e_q(\lambda_1 x)e^{\xi_1}+\alpha_1 e_q(\mu_1 x)e^{\eta_1})(\lambda_1 e_q(\lambda_1 qx)e^{\xi_1}+\alpha_1\mu_1 e_q(\mu_1 qx)e^{\eta_1})}{(\lambda_1 e_q(\lambda_1 x)e^{\xi_1}+\alpha_1(\tau_3)\mu_1 e_q(\mu_1 x)e^{\eta_1})(e_q(\lambda_1 qx)e^{\xi_1}+\alpha_1(\tau_3)e_q(\mu_1 qx)e^{\eta_1})},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{u}_0 &= \frac{1}{w_1}[\tilde{u} - 1 - \tilde{u}\partial_q(w_1)] = \frac{\partial_q^2(h_1)}{\partial_q(h_1)} - \frac{\partial_q(h_1)}{h_1} \\
&= \frac{\lambda_1^2 e_q(\lambda_1 x)e^{\xi_1}+\alpha_1\mu_1^2 e_q(\mu_1 x)e^{\eta_1}}{\lambda_1 e_q(\lambda_1 x)e^{\xi_1}+\alpha_1\mu_1 e_q(\mu_1 x)e^{\eta_1}} - \frac{\lambda_1 e_q(\lambda_1 x)e^{\xi_1}+\alpha_1\mu_1 e_q(\mu_1 x)e^{\eta_1}}{e_q(\lambda_1 x)e^{\xi_1}+\alpha_1 e_q(\mu_1 x)e^{\eta_1}},
\end{aligned}
$$

$$
\tilde{u}_1 = \frac{\tilde{u}_0}{\theta^{-1}(w_1)} = \tilde{u}_0\frac{\theta^{-1}(\partial_q h_1)}{\theta^{-1}(h_1)} = \tilde{u}_0\frac{\lambda_1 e_q(\lambda_1 q^{-1}x)e^{\xi_1}+\alpha_1\mu_1 e_q(\mu_1 q^{-1}x)e^{\eta_1}}{e_q(\lambda_1 q^{-1}x)e^{\xi_1}+\alpha_1 e_q(\mu_1 q^{-1}x)e^{\eta_1}},
$$

$$
\tilde{\phi}_1 = -\dot{\alpha}_1\frac{h_1\partial_q(g_1)-\partial_q(h_1)g_1}{\partial_q(h_1)} = -\frac{d\alpha_1}{d\tau_3}e_q(\mu_1 x)e^{\eta_1}[\mu_1\frac{e_q(\lambda_1 x)e^{\xi_1}+\alpha_1 e_q(\mu_1 x)e^{\eta_1}}{\lambda_1 e_q(\lambda_1 x)e^{\xi_1}+\alpha_1\mu_1 e_q(\mu_1 x)e^{\eta_1}} - 1],
$$

$$
\tilde{\psi}_1 = \theta(\frac{1}{h_1}) = \frac{1}{e_q(\lambda_1 qx)e^{\xi_1}+\alpha_1 e_q(\mu_1 qx)e^{\eta_1}}.
$$

7. Conclusions

In this paper, we generalized the dressing approach for the q-KP hierarchy to the q-KP hierarchy with self-consistent sources (q-KPHSCSs) by combining the dressing method and the method of variation of constants. The usual dressing method for the q-KP hierarchy can not provide the evolution to the new τ_k variable. By introducing some varying constants, say $\alpha(\tau_k)$, we can obtain the desired evolution to τ_k. In this way, we constructed the q-deformed Wronskian solutions to the q-KPHSCSs, and got the exact form for the sources ϕ_i's and ψ_i's.

On the base of eigenfunction symmetry constraint, we constructed a q-mKP hierarchy with self-consistent sources (q-mKPHSCSs) which contains two series of time variables, say t_n and τ_k. The first and second type of q-mKP equation with sources (q-mKPSCS) are obtained as the first two non-trivial equations in the q-mKPHSCSs. And when $q \to 1$ and $\widetilde{u} \equiv 1$, the q-mKPHSCSs reduces to the mKP hierarchy with self-consistent sources [25].

A gauge transformation between the q-KPHSCSs and the q-mKPHSCSs is established in this paper. By using the gauge transformation, we found the Wronskian solutions for the q-mKPHSCSs. The one-soliton solutions to the q-KP equation with a source ((2.6) with $N = 1$) and to the q-mKP equation with a source ((4.5) with $N = 1$) are given explicitly.

It is interesting to consider if there exist solutions in the q-deformed case, which are not surviving limit procedure to the classical case. This will be studied in the future.

Acknowledgments

The authors are grateful to the referees for the quite valuable comments. This work is supported by National Basic Research Program of China (973 Program) (2007CB814800) and National Natural Science Foundation of China (grand No. 10801083 and 10901090). RL acknowledges the economical support from "Banco Santander–Tsinghua University" program for his stay in UCM, and he also thanks the Departamento de Física Teórica II (UCM) for the warm hospitality.

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